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Space–times admitting the complete set of the gauge conditions for higher spin fields

L P Grishchuk† and A D Popova‡

† Shternberg Astronomical Institute, Moscow 117234, USSR
‡ Space Research Institute, Moscow 117810, USSR

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Abstract. We have found and investigated the important class of Riemannian space–times. In these and only in these space–times one can impose the complete set of gauge conditions on massless fields of higher spins. This set reduces the number of independent field variables to two. By this means one can simplify the field equations and interpret the solutions to the equations properly. In this paper we consider curvature of these space–times, their Petrov classification and the uniqueness of the gauge vector field which is used for imposing the complete set of gauge conditions.

1. Introduction

It is known that in flat space–time the equations for the electromagnetic field (spin 1), the spinorial field of spin $\frac{3}{2}$ and the weak gravitational field (spin 2) admit the group of gauge transformations, which leave the equations invariant. The gauge transformations are usually used for imposing the auxiliary (gauge) conditions, which connect the different field variables and reduce the number of independent ones to two. These conditions are similar for all the fields and they imply that certain combinations of field variables $A_{\mu\nu}\ldots$ vanish. They are: the four-divergence $A^{\mu}{}_{\nu\ldots;\mu} = 0$, the trace $A^{\mu}_{\mu\ldots} = 0$ or the similar algebraic combination $\gamma^{\mu}A_{\mu\ldots} = 0$ in the spinorial case and the product of $A_{\mu\nu\ldots}$ with some fixed vector field u^{μ}, $A_{\mu\nu\ldots}u^{\nu} = 0$. For example, the gauge conditions for electromagnetic potentials A_{μ} given in Minkowskian coordinates have the form

$$A^{\mu}{}_{,\mu} = 0, \qquad A_{\mu}u^{\mu} = 0.$$

One usually chooses the vector u^{μ} to be time-like and to have the components $u^{\mu} = (1, 0, 0, 0)$, which means that $A_0 = 0$. In general the vector u^{μ} can be time-like, space-like or null, so we will not restrict ourselves by the time-like u^{μ}. In what follows we call u^{μ} the gauge vector and the mentioned set of gauge conditions a complete set.

The generally covariant equations for fields embedded in curved space–time describe the interaction of these fields with the external gravitational field. These equations for the fields mentioned above possess the group of gauge transformations.

In a curved space–time it is natural to adopt the gauge conditions which are the generally covariant version of the conditions adopted in flat space–time. However, it turns out that in arbitrary curved space–time one cannot use the gauge freedom for imposing the complete set of gauge conditions (Grishchuk and Popova 1981). They may be satisfied jointly only in some limited class of space–times.

The necessary and sufficient condition for this is the existence of the vector field u^μ which obeys the equations (Grishchuk and Popova 1981)

$$u_{\mu;\nu} = u_\mu a_\nu + b g_{\mu\nu}, \tag{1}$$

$$b_{,\mu} - b a_\mu = c u_\mu, \tag{2}$$

where a_μ is an arbitrary vector b and c are arbitrary scalars. In some cases equation (1) exhausts all the restrictions on a space–time metric, so we shall first consider those space–times which admit the vector field (1), and later we shall take into account the additional restriction (2).

Let us start by indicating some properties of the vector field u^μ. According to equation (1) $u_{\mu;\nu}u^\nu$ is proportional to u_μ, which means that the vector field u_μ defines the geodesic congruence. Further, from the same equation it follows that

$$u_\lambda u_{[\mu;\nu]} + u_\mu u_{[\nu;\lambda]} + u_\nu u_{[\lambda;\mu]} = 0,$$

hence, the vector field u^μ differs from a gradient vector field at most by a scalar factor (Petrov 1969).

Let us introduce the norm of u^μ: $u_\mu u^\mu = \pm\rho^2$. By differentiating this equality and by taking into account equation (1) one obtains

$$\rho^2_{,\nu} = 2\rho^2 a_\nu \pm 2 b u_\nu,$$

which leads to $b = 0$ if $\rho = 0$ and

$$a_\nu = (\ln\rho)_{,\nu} \mp b u_\nu$$

if $\rho \neq 0$. It is convenient to carry out the subsequent investigation of equations (1) and (2) separately for two cases: for null u_μ ($u_\mu u^\mu = 0$) and non-null u_μ ($u_\mu u^\mu \neq 0$).

2. Non-null gauge vector

For the vector v^μ with unit norm

$$v_\mu = (1/\rho) u_\mu, \qquad v_\mu v^\mu \equiv \varkappa = \pm 1,$$

equation (1) takes the form

$$v_{\mu;\nu} = p[g_{\mu\nu} - (1/\varkappa) v_\mu v_\nu], \tag{3}$$

where $p = b/\rho$ is an arbitrary scalar. Equation (2) gives the additional restriction

$$p_{,\mu} = (1/\varkappa)(p_{,\alpha} v^\alpha) v_\mu. \tag{4}$$

First, let us show that the non-null vector field (3) may only exist either in flat space–time or in a Ricci non-flat space–time ($R_{\mu\nu} \neq 0$). The integrability conditions for equations (3) have the form

$$\begin{aligned} v^\beta R_{\beta\mu\nu\alpha} = {} & (p^2/\varkappa)(g_{\mu\nu} v_\alpha - g_{\mu\alpha} v_\nu) + p_{,\alpha}[g_{\mu\nu} - (1/\varkappa) v_\mu v_\nu] \\ & - p_{,\nu}[g_{\mu\alpha} - (1/\varkappa) v_\mu v_\alpha], \end{aligned} \tag{5}$$

which yields

$$v^\beta R_{\alpha\beta} = -2 p_{,\alpha} - (1/\varkappa) v_\alpha (p_{,\beta} v^\beta + 3p^2). \tag{6}$$

In a vacuum space–time ($R_{\alpha\beta}=0$) equation (6) would give

$$p_{,\alpha}=-(1/\varkappa)p^2 v_\alpha,$$

which would result in the relation $v^\beta R_{\beta\mu\nu\alpha}=0$. It is known (Zakharov 1972) that if this relation holds along with $R_{\alpha\beta}=0$, then $(v_\lambda v^\lambda)R_{\beta\mu\nu\alpha}=0$ and hence the space–time must be flat since the vector v_α is non-null.

Let us consider the Riemann and Ricci tensors of space-time admitting a gradient vector field (such as the vector field (3), $v_{\mu;\nu}-v_{\nu;\mu}=0$). These tensors can be written down in covariant form as follows.

Introduce the metric tensor $\gamma_{\alpha\beta}$ of the three-dimensional sections orthogonal to the congruence v_α:

$$\gamma_{\alpha\beta}=g_{\alpha\beta}-(1/\varkappa)v_\alpha v_\beta.$$

The tensor $\gamma_\alpha^\beta=\delta_\alpha^\beta-(1/\varkappa)v_\alpha v^\beta$ plays the role of a projection operator. By multiplying an object with the projection operator one can distinguish the components of the object which are orthogonal to v_α. We shall denote the quantities belonging to the orthogonal sections by script letters. Then for the Cristoffel symbols and for the Riemann tensor one has respectively

$$\gamma_\rho^\alpha\gamma_\beta^\sigma\gamma_\gamma^\tau\Gamma_{\sigma\tau}^\rho=\mathcal{T}_{\beta\gamma}^\alpha, \tag{7}$$

$$\gamma_\alpha^\mu\gamma_\beta^\nu\gamma_\gamma^\lambda\gamma_\delta^\rho R_{\mu\nu\lambda\rho}=\mathcal{R}_{\alpha\beta\gamma\delta}+(1/\varkappa)\gamma_\alpha^\mu\gamma_\beta^\nu\gamma_\gamma^\lambda\gamma_\delta^\rho(v_{\mu;\rho}v_{\nu;\lambda}-v_{\mu;\lambda}v_{\nu;\rho}). \tag{8}$$

(For the gradient vector v_α equation (8) establishes a particular case of the general relation derived in Zel'manov (1976).)

If equation (4) is satisfied together with equation (3), then one can introduce the vector field w_α which differs from v_α by a scalar factor and such that equations (3) and (4) take the form

$$w_{\mu;\nu}=qg_{\mu\nu},\qquad q_{,\alpha}=(1/\varepsilon)(q_{,\mu}w^\mu)w_\alpha\equiv dw_\alpha, \tag{9}$$

where $w_\alpha w^\alpha=\varepsilon$ and d is a scalar. A space–time which admits the vector field w_α restricted by equation (9) is called equidistant (Sinukov 1979). For these space–times the relation (8) takes the form

$$\begin{aligned}R_{\mu\nu\rho\sigma}=\mathcal{R}_{\mu\nu\rho\sigma}&+(d/\varepsilon)(g_{\nu\rho}w_\mu w_\sigma-g_{\nu\sigma}w_\mu w_\rho\\&+g_{\mu\sigma}w_\nu w_\rho-g_{\mu\rho}w_\nu w_\sigma)+(q^2/\varepsilon)(\gamma_{\nu\rho}\gamma_{\mu\sigma}-\gamma_{\mu\rho}\gamma_{\nu\sigma}),\end{aligned} \tag{10}$$

where equation (9) has been used. From equation (10) it follows that

$$R_{\mu\nu}=\mathcal{R}_{\mu\nu}-d\left(g_{\mu\nu}+\frac{2}{\varepsilon}w_\mu w_\nu\right)-\frac{2q^2}{\varepsilon}\left(g_{\mu\nu}-\frac{1}{\varepsilon}w_\mu w_\nu\right). \tag{11}$$

The tensors $\mathcal{R}_{\mu\nu\lambda\rho}$ and $\mathcal{R}_{\mu\nu}$ determine the geometry of sections orthogonal to w_α and they are unrestricted.

Equation (11) gives certain limitations on the general structure of the energy-momentum tensor, which could govern the equidistant space–time according to the Einstein equations.

Let us give the Petrov classification of the equidistant space–times. It can be given fairly completely even without using the exact form of the metric tensor. Multiply equation (10) and equation (11) by w^μ. One gets respectively

$$w^\mu R_{\mu\nu\rho\sigma} = d(g_{\nu\rho} w_\sigma - g_{\nu\sigma} w_\rho), \tag{12}$$

$$w^\mu R_{\mu\nu} = -3 d w_\nu. \tag{13}$$

(Equation (12) can also be derived as the integrability condition for equation (9).) By using the expression for the Weyl tensor and taking into account equations (12), (13), one finds

$$w^\mu C_{\mu\nu\rho\sigma} = \tfrac{1}{2}(w_\sigma R_{\nu\rho} - w_\rho R_{\nu\sigma}) - \tfrac{1}{2}(d + \tfrac{1}{3}R)(w_\sigma g_{\nu\rho} - w_\rho g_{\nu\sigma}). \tag{14}$$

It is important to notice that if the vector ξ_α is an eigenvector of the Ricci tensor† $\xi^\beta R_{\alpha\beta} = \lambda \xi_\alpha$, then the simple bivector $W^{\alpha\beta} = w^\alpha \xi^\beta - w^\beta \xi^\alpha$ is an eigen-bivector of the Weyl tensor. Indeed, equation (14) yields

$$W^{\mu\nu} C_{\mu\nu\rho\sigma} = \Lambda W_{\rho\sigma}, \qquad \Lambda = d + \tfrac{1}{3}R - \lambda. \tag{15}$$

According to equation (13), one of the eigenvectors of $R_{\mu\nu}$, and moreover the real one, is w^μ.

Let w^μ be a time-like vector. Then the other eigenvectors, orthogonal to w^μ, must be space-like and also real (see, for example, Landau and Lifshitz 1971). Since the tensor $\mathscr{R}_{\mu\nu}$ is arbitrary it has, in general, three different space-like eigenvectors. Denote them by $\overset{(a)}{\xi}{}^\alpha$ $(a = 1, 2, 3)$. Then $\overset{(a)}{W}{}_\alpha{}^\beta$ are simple and real eigen-bivectors of the Weyl tensor. Moreover, the eigenvalues $\overset{(a)}{\Lambda}$ satisfy equation $\Sigma_{a=1}^3 \overset{(a)}{\Lambda} = 0$, which follows from equation (15) and from the fact that for the non-null eigenvectors of $R_{\mu\nu}$ the following equality holds: $R = \Sigma_{a=1}^3 \overset{(a)}{\lambda} + \overset{(o)}{\lambda}$ where $\overset{(o)}{\lambda} = -3d$. Thus, in the general case we have three different real eigen-bivectors of the Weyl tensor, which means that the metric belongs to type I with the real eigenvalues. If two or three eigenvalues $\overset{(a)}{\Lambda}$ coincide, then the type of the space–time reduces to type D or type O. The latter case takes place if the Ricci tensor simplifies to the form

$$R_{\mu\nu} = \alpha w_\mu w_\nu + \beta g_{\mu\nu}.$$

If w^μ is a space-like vector then the eigenvectors of the Ricci tensor can be real, complex or null. This means that the eigen-bivectors of the Weyl tensor can be complex conjugate or null. According to this, among the equidistant space–times appear representatives of types II, N and III.

Now let us consider the problem of uniqueness of the gauge vector w_μ. First, we shall show that there exists only one gauge vector w_μ if $d \neq$ constant. Suppose that there are two different (non-collinear) vectors w^μ and $\bar{w}^\mu$, $\bar{w}^\mu \neq \alpha w^\mu$. For $\bar{w}^\mu$, as well as for w^μ, equation (9) is valid and the equation similar to equation (12) should also be satisfied:

$$\bar{w}^\mu R_{\mu\nu\rho\sigma} = \bar{d}(g_{\nu\rho} \bar{w}_\sigma - g_{\nu\sigma} \bar{w}_\rho). \tag{16}$$

Multiplying equation (16) by w^μ and multiplying equation (12) by $\bar{w}^\mu$ and then adding one to another, one gets the relation $d = \bar{d}$. It now follows from the relation $q_{,\alpha\beta} - q_{,\beta\alpha} = 0 = \bar{q}_{,\alpha\beta} - \bar{q}_{,\beta\alpha}$ that either $\bar{w}^\mu$ is proportional to w^μ, which contradicts the initial suggestion, or $d =$ constant.

In the case $d =$ constant the number of different gauge vectors can be larger than one (this is obvious if $d = 0$).

† Remember that $R_{\alpha\beta} \neq 0$ except for the trivial case $R_{\mu\nu\alpha\beta} = 0$.

3. Null gauge vector

Since in this case $b=0$, the equality (2) gives $c=0$ and does not bring any new restrictions on the gauge vector. The only restriction is equation (1). Introduce $v_\mu=(\exp\alpha)e_\mu$ where $e_{[\mu;\nu]}=0$. By alternating indices μ and ν in equation (1) one sees that $a_\mu-\alpha_{,\mu}$ is proportional to u_μ. This allows one to rewrite equation (1) in the form

$$e_{\mu;\nu}=me_\mu e_\nu, \tag{17}$$

where m is an arbitrary scalar. The integrability conditions for equation (17) are

$$e^\mu R_{\cdot\mu\nu\alpha\beta}=e_\nu(m_{,\alpha}e_\beta-m_{,\beta}e_\alpha). \tag{18}$$

Equation (18) leads to

$$e^\mu R_{\mu\nu}=(m_{,\alpha}e^\alpha)e_\nu,$$

so that the vector e^μ is an eigenvector of the Ricci tensor.

In order to write down the formulae for the Riemann and Ricci tensors of a space–time subjected to equation (17), we will need the relation between the Riemann tensor of the space–time and the Riemann tensor of the two-dimensional space V_2 orthogonal to the two given geodesic congruences. Let n^μ and t^μ be time-like and space-like gradient vectors, respectively. We choose them to be orthogonal and have unit norms:

$$n_\mu t^\mu=0,\qquad n_\mu n^\mu=1,\qquad t_\mu t^\mu=-1.$$

Applying equations (7) and (8) twice, one obtains

$$\gamma'^\mu_\alpha\gamma'^\nu_\beta\gamma'^\lambda_\gamma\gamma'^\rho_\delta R_{\mu\nu\lambda\rho}=\mathcal{R}'_{\alpha\beta\gamma\delta}+\gamma'^\mu_\alpha\gamma'^\nu_\beta\gamma'^\lambda_\gamma\gamma'^\rho_\delta(t_{\nu;\lambda}t_{\mu;\rho}-t_{\mu;\lambda}t_{\nu;\rho}-n_{\nu;\lambda}n_{\mu;\rho}+n_{\mu;\lambda}n_{\nu;\rho}), \tag{19}$$

where

$$\gamma'^\alpha_\beta=\delta^\alpha_\beta-n^\alpha n_\beta+t^\alpha t_\beta,$$

and $\mathcal{R}'_{\alpha\beta\gamma\delta}$ is the Riemann tensor of the two-dimensional space. For every V_2 the following relation is valid:

$$\mathcal{R}'_{\alpha\beta\gamma\delta}=\mathcal{K}(\gamma'_{\alpha\gamma}\gamma'_{\beta\delta}-\gamma'_{\alpha\delta}\gamma'_{\beta\gamma}).$$

Now introduce e^μ and $\bar e^\mu$ as follows:

$$e^\mu=(1/\sqrt{2})(n^\mu+t^\mu),\qquad \bar e^\mu=(1/\sqrt{2})(n^\mu-t^\mu).$$

Let the null vector e^μ satisfy equation (17). Then from equation (19) one obtains

$$\begin{aligned}R_{\mu\nu\lambda\rho}=\mathcal{K}(\gamma'_{\mu\lambda}\gamma'_{\nu\rho}-\gamma'_{\mu\rho}\gamma'_{\nu\lambda})+\bar e^\alpha(e_\mu R_{\alpha\nu\lambda\rho}-e_\nu R_{\alpha\mu\lambda\rho}+e_\lambda R_{\alpha\rho\mu\nu}-e_\rho R_{\alpha\lambda\mu\nu})\\ -\bar e^\alpha\bar e^\beta(e_\mu e_\rho R_{\alpha\nu\beta\lambda}-e_\mu e_\lambda R_{\alpha\nu\beta\rho}+e_\nu e_\lambda R_{\alpha\mu\beta\rho}-e_\nu e_\rho R_{\alpha\mu\beta\lambda}).\end{aligned}$$

As a consequence of this equation one has

$$R_{\mu\nu}=e_\mu A_\nu+e_\nu A_\mu+\mathcal{K}g_{\mu\nu}, \tag{20}$$

where

$$A_\mu=R_{\mu\nu}\bar e^\nu-\tfrac{1}{2}e_\mu R_{\alpha\beta}\bar e^\alpha\bar e^\beta-\mathcal{K}\bar e_\mu,$$

$$\mathcal{K}=(A_\nu-m_{,\nu})e^\nu.$$

Equation (20) determines the structure of the energy–momentum tensor which can govern the dynamics of space–time.

Let us give the Petrov classification of the space–time under consideration. By using equations (18) and (20) one gets

$$
\begin{aligned}
e^{\alpha}C_{\alpha\beta\gamma\delta} &= e_{\beta}[e_{\gamma}(m_{,\delta}-\tfrac{1}{2}A_{\delta})-e_{\delta}(m_{,\gamma}-\tfrac{1}{2}A_{\gamma})]+\tfrac{1}{3}(e_{\delta}g_{\beta\gamma}-e_{\gamma}g_{\beta\delta})(m_{,\alpha}-A_{\alpha})e^{\alpha}, \\
e^{\alpha}e^{\gamma}C_{\alpha\beta\gamma\delta} &= \tfrac{1}{3}e_{\beta}e_{\delta}e^{\alpha}(A_{\alpha}-2m_{,\alpha}) = -\tfrac{1}{6}Re_{\beta}e_{\delta}, \qquad (21)\\
e^{\alpha}e^{\gamma}C_{\alpha\beta\gamma[\delta}e_{\lambda]} &= 0.
\end{aligned}
$$

It follows from equation (21) that the space–time cannot belong to type I. A further corollary is that if $R\neq 0$ the space–time must belong to type II or type D. It follows from equation (21) that the eigenvalues $(\alpha+\mathrm{i}\beta)_{1,2,3}$ in cases II and D look like

$$\alpha_1 = -\tfrac{1}{6}R, \qquad \alpha_2=\alpha_3=\tfrac{1}{12}R, \qquad \beta_1=\beta_2=\beta_3=0,$$

when $R=0$, and as a special case in vacuum space–times ($R_{\mu\nu}=0$) the Weyl tensor can be of types III, N and O only. (It should be emphasised that the covariantly constant vector field e_α may exist only in space–times of types N and O.)

Now let us turn to the problem of the uniqueness of the gauge vector e_α. It is seen from equation (21) that e_α is the degenerate null eigenvector of the Weyl tensor. Because of that the vector e_α is unique except, may be, for the space–times of type D where there exist two degenerate null vectors and type O where there exist many gauge vectors.

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